Path-integral treatment of optical and microwave tunneling

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A model based on a statistical method, as deduced from a path-integral treatment of the Brownian motion proposed by Feynman and Hibbs in 1965, demonstrates to be capable of interpreting the results of delay time measurements in frustrated total internal reflection experiments at an optical and microwave scale. A plausible description of the trajectories followed by the system inside the tunneling region, the air gap between the two dielectric prisms, is given.

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As is well-known, if it is difficult to give an unambiguous definition of tunneling time, it is even more difficult to obtain information regarding the paths adopted by the "system" while tunneling or, in other words, to give a description of the trajectories (in a semiclassical picture) followed by the system in these processes.

According to authoritative and widely shared opinions, this problem appeared not to have a solution since, even within the semiclassical approaches, there are no simultaneous (real) solutions to the Euler-Lagrange equation and the Hamilton-Jacobi equation. Some progress has been made by adopting complex-valued functions that describe the motion of the "particle." This satisfies the above-mentioned equations in a path-integral treatment over complex paths, and thereby provides a full analog for the path integral of the stationary phase approximation [1].

An alternative approach to this problem can be envisaged along the lines of a stochastic modeling of the telegrapher's equation [2], according to which the system follows zigzag paths in space time. These are particularly suitable for describing unidimensional problems [3].

In this paper we shall follow a different procedure, by adopting and developing a statistical method as earlier proposed by Feynman and Hibbs for the Brownian motion [4] and adapting it to the tunneling, i.e., to the cases of frustrated total internal reflection (FTIR) of microwaves [5] and light [6]. In this way, we can obtain information about the shape of the trajectories in the bidimensional space of the air-gaps that separate two dielectric prisms.

The experimental results of traversal time versus the gap width relative to the two cases considered, as taken from Refs. [5] and [6], are reported in Figs. 1(a) and 1(b), respectively. They exhibit a more or less pronounced structure which can be attributed to a coarse grain of the optical paths. As for the curves fitting these data, see below for the results of our analysis.

Alternative stochastical model. By analogy with the stochastical process experienced by a particle traversing a material slab (Brownian motion) as treated in Ref. [4], we assume that the tunneling process in the air gap has a stochastic nature that is similar to the mechanism adopted in Ref. [2].

Let us use p to denote the probability that the "particle" suffers a "collision," with N being the total number of collision centers traveling across the gap of width T. Therefore $\bar{n}=pN$ is the average number of collisions, and $\mu=\bar{n}/T$ is the average number per unitary length. Analogously, $1-p$ is the probability of the absence of collisions. The probability that the particle suffers n collisions (avoiding $N-n$) is given by [7]

$$P_n=\binom{N}{n}p^n(1-p)^{N-n}\simeq\frac{e^{-\bar{n}}\bar{n}^n}{n!},\tag{1}$$

where the last member holds true for $p\ll 1$ and $N\gg 1$.

This picture represents what can be considered as the fine grain of the paths. The deviation angle from the incoming direction is given by

$$\theta(t)=\sum_{j=1}^{n}a_j\eta(t-t_j),\tag{2}$$

where a_j is the deflection after the jth collision and η is the step function. The deviations occur with variable intensities distributed according to a Gaussian $p(a)=(2\pi\sigma)^{-1/2}\exp(-a^2/2\sigma^2)$ with a standard deviation of σ^2. The transversal displacement dx, which corresponds to the longitudinal one dt, is $dx=\theta dt$, $\dot{x}=\theta$, and $\ddot{x}=\dot{\theta}$. By differentiating Eq. (2), we have

$$f(t)=\dot{\theta}(t)=\sum_{j=1}^{n}a_j\delta(t-t_j)\tag{3}$$

and the probability functional corresponding to the "history" of deviations is given as it follows.

Probability functional. This section is devoted to a schematic description of the obtainment of the probability functional, which represents the crucial point in the adopted modeling.

By starting from the definition of "characteristic function" $\Phi(k)$ of a probability distribution $P(x)$ [4], it is natural to extend these definitions to the case of distributions of func-

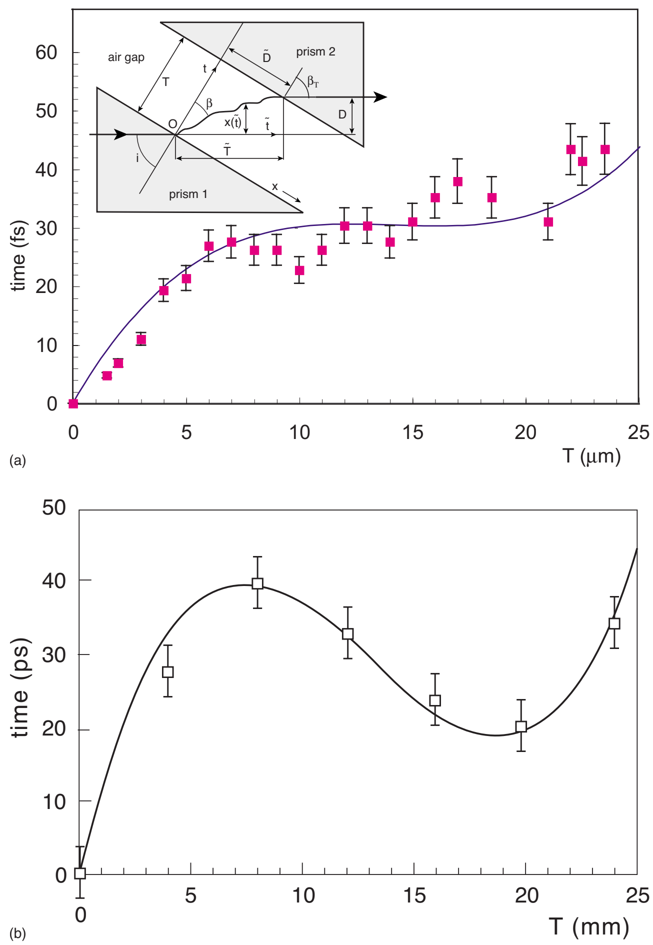

FIG. 1. (Color online) Traversal time results, as a function of the gap width between the two prisms in FTIR experiments whose geometry is sketched in the inset. For optical laser beam (a): wavelength $\lambda=3.39$ μm, $i=45.5°$, and $\rho=1.409$; for the microwave case (b): $\lambda=3.2$ cm, $i=60°$, and $\rho=1.49$. The curves are obtained from Eq. (21) with parameter values as, for (a): $T(\max)=23.7$ μm, $D=12.1$ μm, and $\beta_T=22.3°$; and for (b): $T(\max)=24$ mm, $D=22.6$ mm, and $\beta_T=40.5°$.

tions and not of single variables or discrete vectors. Therefore $P(x)$ becomes a functional $P[f(t)]$, and we obtain formally

$$\Phi[k(t)] = \int \mathcal{D}f(t) e^{i\int_0^T dt k(t) f(t)} P[f(t)], \tag{4}$$

$$P[f(t)] = \int \mathcal{D}k(t) e^{-i\int_0^T dt k(t) f(t)} \Phi[k(t)], \tag{5}$$

where $\Phi[k(t)]$ is the "characteristic functional" of $P[f(t)]$. These equations are formal ones, since the rule for their effective calculation is not given. $\mathcal{D}$ is the symbol of functional integration, and $P[f(t)]\mathcal{D}f(t)$ is the probability that the function $f(t)$ lies in a neighbor $\mathcal{D}f(t)$. However, a precise meaning to these words can be given only by a "discretization" procedure [8]: by dividing the interval $(0,T)$ by N points t_j, when $N\to\infty$, $P\mathcal{D}f$ is the probability that $f(t)$ lies between f_1 and f_1+df_1 at $t=t_1$, between f_2 and f_2+df_2 at $t=t_2,\ldots$ between f_N and f_N+df_N at $t=t_N$ $[f_j=f(t_j)]$.

We will return to Eq. (5) later. Our aim now is to calculate $\Phi[k(t)]$ directly in a heuristic way, and then to use Eq. (5) in order to obtain the probability.

We are interested in a stochastic function $f(t)$, i.e., a function that represents a random process; in particular, a class of functions of the form

$$f(t) = \sum_{j=1}^{n} a_j u(t-t_j), \quad 0 < t_1 < t_2 < \cdots < t_n < T \tag{6}$$

that represents a sequence of n pulses distributed at random points t_j in the spatial (or time) interval $(0,T)$, and that have intensities a_j and the same normalized shape $u(t)$. Let us assume that the number n of pulses is distributed with probability P_n, and that the intensities a_j are distributed with a probability density $p(a)$. From Eq. (4) it follows that

$$\Phi[k(t)] = \int \mathcal{D}f(t) e^{i\int_0^T dt k(t) f(t)} P[f(t)]$$
$$= \sum_{n=0}^{\infty} P_n \prod_{j=1}^{n} \int_{-\infty}^{\infty} da_j \exp\left[i a_j \int_0^T dt k(t) u(t-t_j) \right] p(a_j). \tag{7}$$

We would like to point out that the calculation of Eq. (7) represents the "heuristic" step in the procedure by reducing the computation of a functional integral to a sort of mean value. Defining

$$W[\omega] = \int_{-\infty}^{\infty} da\, e^{i\omega a} p(a) \tag{8}$$

and assuming that the n pulses are distributed with uniform probability over the complete spatial (or time) interval, we are led to write

$$\Phi[k(t)] = \sum_{n=0}^{\infty} P_n \left(\frac{\gamma}{T} \right)^n,$$

where

$$\gamma = \int_0^T ds W\left[\int_0^T dt k(t) u(t-s) \right]. \tag{9}$$

If the number of pulses $(0,T)$ obeys a Poisson distribution, Eq. (1), and by recalling that $\bar{n}=\mu T$, we have

$$\Phi[k(t)] = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\gamma \bar{n}}{T} \right)^n = e^{-\mu(T-\gamma)}. \tag{10}$$

By admitting that the pulse shape is very narrow, i.e., $f(t)$ has the form (3), it follows that

$$W\left[\int_0^T dt k(t) u(t-s) \right] = W[k(s)] \tag{11}$$

and

$$\Phi[k(t)] = \exp\left(-\mu\int_0^T dt\{1 - W[k(t)]\}\right). \tag{12}$$

If the distribution of intensities is Gaussian with zero mean and standard deviation σ^2, Eq. (8) gives $W[\omega] = \exp(-\sigma^2\omega^2/2)$ and, for small values of σ, $\Phi[k(t)]$ takes the form

$$\Phi[k(t)] = \exp\left(-\frac{\mu\sigma^2}{2}\int_0^T dt[k(t)]^2\right). \tag{13}$$

Now, from the characteristic functional $\Phi[k(t)]$, we can obtain the probability functional $P[f(t)]$ from Eq. (5). By using the discretization procedure and introducing a suitable normalization constant B,

$$P[f(t)] = \lim_{N\to\infty}\prod_{j=1}^{N} B\int_{-\infty}^{\infty} dk_j e^{-\Delta t(ik_jf_j + bk_j^2)}, \tag{14}$$

where $b=\mu\sigma^2/2$. The calculation is a standard one [8], and the final result is just

$$P[f(t)] = \exp\left(-\frac{1}{2R}\int_0^T dt[f(t)]^2\right), \tag{15}$$

where $R=\mu\sigma^2$. Since there is a biunivocal correspondence between the trajectory $x(t)$ and $f(t)$, $P[x(t)] \propto P[f(t)]$. By recalling that $\ddot{x}(t)=f(t)$, from Eq. (15), we obtain

$$P[x(t)] \propto \exp\left(-\frac{1}{2R}\int_0^T dt[\ddot{x}(t)]^2\right). \tag{16}$$

We would like to point out the difference between this expression and the one obtained for the usual Brownian motion described by the diffusion constant d, giving the probability of a Brownian path [9]:

$$P[x(t)] \propto \exp\left(-\frac{1}{4d}\int_0^T dt[\dot{x}(t)]^2\right). \tag{17}$$

Our type of Brownian motion, on the contrary, is a somewhat particular one, driven by a Poisson process and with a well-defined initial velocity [4]. It follows that the probability functional is different, and it is given by Eq. (16).

Application to our problem. We are interested in the probability $P(D,\psi)$ that we have $x(T)=D$, $\theta(T)=\psi$, the initial conditions being $x(0)=0$, $\theta(0)=0$. Therefore we have to integrate $P[x(t)]$ over all the trajectories satisfying the above boundary conditions, namely, by functional integration,

$$P(D,\psi) \propto \int_{x(0)=0,\ \theta(0)=0}^{x(T)=D,\ \theta(T)=\psi} \mathcal{D}x(t)\exp\left(-\frac{1}{2R}\int_0^T dt[\ddot{x}(t)]^2\right). \tag{18}$$

This integral, in the limit of small R, is dominated by the trajectories that minimize the functional in the exponent. We know that, if the functional has the form $\int_0^T dt F[\ddot{x}(t)]$, the extremal is supplied by the Euler-Poisson equation [10] $(d^2/dt^2)(\partial F/\partial\ddot{x})=0$ which, in our case, gives $\dddot{x}=0$. By using the boundary conditions, we easily obtain [11]

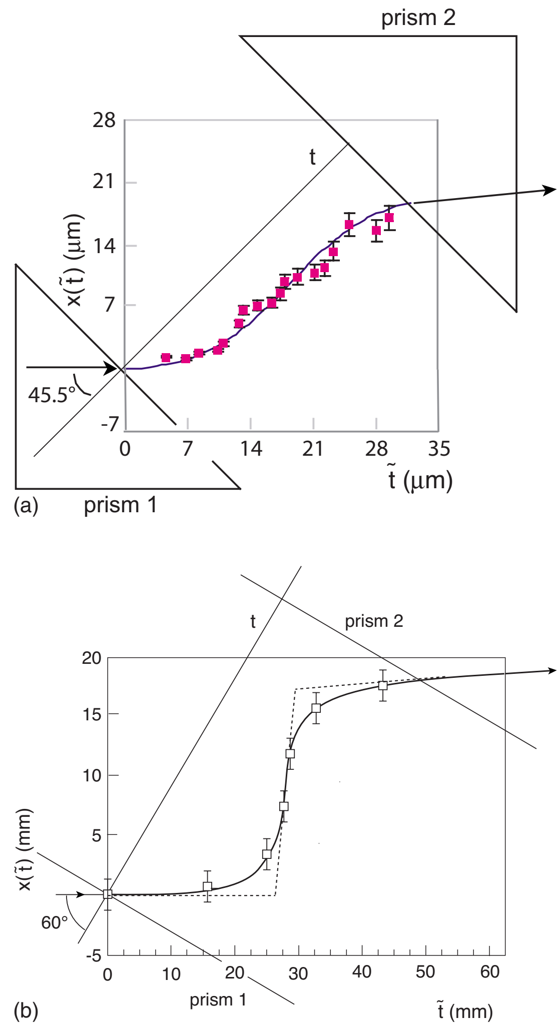

FIG. 2. (Color online) Most probable paths inside the air gap in the optical case (a) and microwave case (b). The continuous curves which fit the data of Figs. 1(a) and 1(b), respectively, are given by the function $x(\tilde{t})$, when situated in the gap space. The dotted line in (b) resembles a typical stochastic path [3].

$$x(t) = (3D - \psi T)\left(\frac{t}{T}\right)^2 + (\psi T - 2D)\left(\frac{t}{T}\right)^3. \tag{19}$$

This equation resolves our problem, since it will supply the shape of the most probable paths in the gap space. It is worth observing that the corresponding path in the case of the probability expressed by Eq. (17), in the limit of small d, is given by the equation $\ddot{x}=0$ (free particle).

We first want to derive the expression of the traversal time that describes the data reported in Fig. 1. According to the geometry adopted [see the inset in Fig. 1(a)], the angle θ will be replaced by $i-\beta$, where i is the incidence angle and β is the mean deviation angle with respect to the perpendicular of the gap. Analogously, ψ becomes $i-\beta_T$. In this a way, T becomes $\tilde{T}=T\cos(i-\beta_T)/\cos\beta_T$ and t becomes $\tilde{t}=t\cos(i-\beta)/\cos\beta$.

In order to simplify the analysis, let us assume that $t/T \simeq \tilde{t}/\tilde{T}$. According to Refs. [5] and [6], the quantity $\tilde{D}$ in

Fig. 1 is a measure of the traversal time which is given by $\tau=\rho\tilde{D}/c\sin i$, with $c\sin i/\rho$ being the velocity component along the gap of the incident ray and ρ the refractive index of the prisms. By taking into account these relations, the traversal time as a function of t can be expressed as

$$\tau(t)=\frac{2\rho}{c\sin(2i)}[t\sin i-x(t)], \tag{20}$$

where $x(t)$ is now given by

$$x(t)=[3D-(i-\beta_T)\tilde{T}]\left(\frac{t}{T}\right)^2+[(i-\beta_T)\tilde{T}-2D]\left(\frac{t}{T}\right)^3. \tag{21}$$

The quantities β_T and D in this last expression should be considered as moderately adjustable parameters so that they fit the results in Figs. 1(a) and 1(b), as shown by the curves reported there which rather well reproduce the behavior of the experimental data [12].

When properly located in the space of the gaps, the function $x(\tilde{t})$ gives a plausible description of the most probable trajectories followed by the "system" in the gap between the two prisms. The results relative to the case of optical and microwave tunneling are shown in Figs. 2(a) and 2(b), respectively. From an inspection of these figures, we note that, apart from a realistic and plausible rounding, these curves are reminiscent of the zigzag paths hypothesized in the model of Ref. [3]. The step dimensions of what we consider to be the coarse grain of the paths are, for the case of a microwave of the order of centimeters, analogous with the findings of Ref. [3]. In the case of optical tunneling, these dimensions are obviously smaller, of the order of microns, but in any case are comparable with the wavelength. As mentioned before, the fine structure of the paths, the dimensions (dx,dt) of which can be assumed to be the average interval between successive diffusive processes, is presumably several orders of magnitude smaller than the one relative to the coarse grain.

We can therefore conclude that the statistical method adopted, although inspired from Ref. [4], represents the main core of the approach to our problem, and deserves to have the schematic description as given before. By utilizing it, we were able to obtain an alternative interpretation of tunneling in FTIR experiments [13], according to a procedure which can be considered to be an extension to two-dimensional cases of the fecund approach originally proposed by Kac, as mentioned in Ref. [2]. In this way we have obtained, besides the description of the traversal time of the gap (which constitutes the barrier in FTIR), the shape of the most probable trajectories (rays) inside the tunneling region.

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